Externally Definable Sets and Shelah Expansions

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September 22, 2016

Set Up and Notation

Let $\mathcal{L}$ be a language.

Let T be a complete $\mathcal{L}$-theory with an infinite model $\mathcal{M}$.

Let $\mathcal{U}$ denote the monster model of T.

We will view all models of T as elementary substructures of $\mathcal{U}$.

We will let x, y, z, ... range over finite tuples of variables and a, b, c, ... over finite tuples of parameters.

Set Up and Notation

Suppose $B \subset U$.

We will use $\mathcal{L}(B)$ to denote the set of all $\mathcal{L}$-formulae with parameters in B; i.e.,

$$\mathcal{L}(B) = \{\phi(x, b) : \phi(x, y) \in \mathcal{L} \text{ and } b \in B^{|y|}\}.$$

Given $a \in U$, we will use $\mathrm{tp}(a/B)$ to denote the "type of a over B"; i.e.,

$$\mathrm{tp}(a/B) = \{\phi(x, b) \in \mathcal{L}(B) : \mathcal{U} \models \phi(a, b)\}.$$

We will use $S_n(B)$ to denote the set of all complete n-types over B; i.e.,

$$S_n(B) = \{\mathrm{tp}(a/B) : a \in U^n\}.$$

Traces and Induced Structures

Let $A \subset U$, $\phi(x, y) \in \mathcal{L}$, and $b \in U$.

Definition

The *trace* of $\phi(x, b)$ in A is

$$\phi(A, b) = \{a \in A^{|x|} : \mathcal{U} \models \phi(a, b)\}.$$

We can induce a structure on A using traces.

Definition

Given $B \subset U$, define the language

$$\mathcal{L}_{\mathrm{ind}B} = \{R_{\phi(x,b)} : \phi(x, b) \in \mathcal{L}(B)\}$$

and let $A_{\mathrm{ind}B}$ denote the structure with domain A such that for all $a \in A^{|x|}$, we have

$$A_{\mathrm{ind}B} \models R_{\phi(x,b)}(a) \iff \mathcal{U} \models \phi(a, b).$$

Externally Definable Sets and Shelah Expansions

Definition

We call $X \subseteq M^n$ *externally definable* iff:
there exists $\phi(x, y) \in \mathcal{L}$ and $b \in U$ such that $X = \phi(M, b)$.

Let $\mathcal{M}' \succ \mathcal{M}$ be $|M|^+$-saturated.

Let $\mathcal{L}^{\mathsf{Sh}} = \mathcal{L}_{\mathrm{ind}M'} = \{R_{\phi(x,b)} : \phi(x, b) \in \mathcal{L}(M')\}$.

Let $\mathcal{M}^{\mathsf{Sh}} = M_{\mathrm{ind}M'}$.

By saturation, $\mathcal{M}^{\mathsf{Sh}}$ contains a predicate for every externally definable subset of M.

We will show that if T is NIP, then $\mathcal{M}^{\mathsf{Sh}}$ has quantifier elimination (QE).

Why do we care?

For any $A, B \subset U$, let $\text{Traces}(A, B)$ denote the collection of all traces in A by formulae with parameters in B.

For any structure $\mathcal{A}$, let $\mathcal{D}(\mathcal{A})$ denote the collection of all sets definable in $\mathcal{A}$ by formulae with parameters in A.

In general:

- $\text{Traces}(A, B) \subseteq \mathcal{D}(A_{\text{ind}B})$
- $\text{Traces}(M, M') = \text{Traces}(M, U) \subseteq \mathcal{D}(\mathcal{M}^{\text{Sh}})$

If $\mathcal{M}^{\text{Sh}}$ has QE:

- $\text{Traces}(M, M') = \text{Traces}(M, U) = \mathcal{D}(\mathcal{M}^{\text{Sh}}) = \mathcal{D}((\mathcal{M}^{\text{Sh}})^{\text{Sh}})$

Why do we care?

Easy way to generate weakly o-minimal structures:

- If T is o-minimal (e.g., DLO, ODAG, RCF), it follows that $\mathcal{M}^{\mathrm{Sh}}$ is weakly o-minimal.

Current Research:

- What conditions are sufficient for $M_{\mathrm{ind}A}$ to have QE?

Heirs and Coheirs

Suppose $M \subseteq B \subset U$. Let $q(x) \in S(B)$ extend $p(x) \in S(M)$.

Definition

We say q is an *heir* of p iff: q "satisfies no new formulae," meaning

$$\phi(x, b) \in q \quad \Longrightarrow \quad \text{for some } m \in M, \quad \phi(x, m) \in p.$$

Intuition: The heirs of a type are the extensions of that type that are most like the original.

Definition

We say q is a *coheir* of p iff: q is finitely satisfiable in M.

Fact: Types over models have heirs and coheirs over any larger set of parameters.

Heir/Coheir Duality

For $a, b \in U$, TFAE:

- $\mathrm{tp}(a/Mb)$ is an heir of $\mathrm{tp}(a/M)$
- $\mathrm{tp}(b/Ma)$ is a coheir of $\mathrm{tp}(b/M)$
- for all $\phi(x, y) \in \mathcal{L}$, if $\mathcal{U} \models \phi(a, b)$, then $\mathcal{U} \models \phi(a, m)$ for some $m \in M$

Example: $(\mathbb{R}, <) \succ ((-1, 1), <) \models \mathrm{DLO}$

- $\mathrm{tp}(3/(-1, 1) \cup \{2\})$ is an heir but not a coheir of $\mathrm{tp}(3/(-1, 1))$
- $\mathrm{tp}(2/(-1, 1) \cup \{3\})$ is a coheir but not an heir of $\mathrm{tp}(2/(-1, 1))$

Coheir Sequences are Indiscernible

Suppose $M \subseteq B \subset U$ and $q(x) \in S(B)$ is finitely satisfiable in M.

(Note: q is a coheir of $q\restriction_M$)

Definition

A sequence $(b_i : i < \omega) \subseteq B$ such that $b_i \models q\restriction_{Mb_{<i}}$ is called a *coheir sequence* for q over M.

Lemma

Coheir sequences over M are indiscernible over M.

Coheir Sequences are Indiscernible

Proof: Suppose $M \subseteq B \subset U$. Let $q(x) \in S(B)$ be finitely satisfiable in M. Suppose $(b_i : i < \omega) \subseteq B$ and $b_i \models q\restriction_{Mb_{<i}}$.

Let $P(n)$ denote the following assertion: $\forall\, i_1 < \cdots < i_n \,\forall\, \phi \in \mathcal{L}(M)$

$$\mathcal{U} \models \phi(b_{i_1}, ..., b_{i_n}) \leftrightarrow \phi(b_1, ..., b_n).$$

Assume $\neg P(n+1)$. So $\exists\, i_1 < \cdots < i_{n+1} \,\exists\, \phi \in \mathcal{L}(M)$

$$\mathcal{U} \models \phi(b_{i_1}, ..., b_{i_n}, b_{i_{n+1}}) \wedge \neg\phi(b_1, ..., b_n, b_{n+1}).$$

It follows that

$$\phi(b_{i_1}, ..., b_{i_n}, x), \quad \neg\phi(b_1, ..., b_n, x) \quad \in \quad q.$$

Since q is finitely satisfiable in M, there exists $m \in M$ such that

$$\mathcal{U} \models \phi(b_{i_1}, ..., b_{i_n}, m) \wedge \neg\phi(b_1, ..., b_n, m)].$$

But this implies $\neg P(n)$, so the lemma holds by induction on n.

The Independence Property

Definition

We say that T has the *independence property (is IP)* iff: for some $\phi(x, y) \in \mathcal{L}$, there exist sequences of parameters $(a_n : n < \omega)$ and $(b_X : X \subseteq \omega)$ such that

$$\mathcal{U} \models \phi(a_n, b_X) \iff n \in X.$$

Fact: T is IP if and only if for some $\phi(x, u) \in \mathcal{L}(U)$, there exists a sequence of parameters $(a_n : n < \omega)$ which is indiscernible over $\varnothing$ such that

$$\mathcal{U} \models \phi(a_n, u) \iff n \text{ is even}.$$

Definition

We say that T is NIP iff: T is not IP.

Notation for the Quantifier-Free Setting

We will use "qf" as a subscript when we wish to consider only quantifier-free formulae. For example, given $a \in U$ and $B \subset U$:

- $\mathcal{L}_{\mathrm{qf}}(B)$ denotes the quantifier-free formulae in $\mathcal{L}(B)$
- $S_{\mathrm{qf}}(B)$ denotes the complete quantifier-free types over B
- $\mathrm{tp}_{\mathrm{qf}}(a/B)$ denotes the quantifier-free type of a over B

Quantifier-Free-Definable Types

Definition

We say that $p(x) \in S_{\text{qf}}(B)$ is *quantifier-free definable* iff: for every $\phi(x, y) \in \mathcal{L}_{\text{qf}}$, there exists $d_\phi(y) \in \mathcal{L}_{\text{qf}}(B)$ such that for all $b \in B^{|y|}$, we have

$$\phi(x, b) \in p \iff \mathcal{U} \models d_\phi(b).$$

In such cases, we call $d = \{d_\phi : \phi \in \mathcal{L}_{\text{qf}}\}$ a *defining schema* for p.

Fact: If $A \subset U$, then $d(A) = \{\phi(x, a) : \mathcal{U} \models d_\phi(a)\} \in S_{\text{qf}}(A)$.

Example: $(\mathbb{Q}, <) \models \text{DLO}$

- $\text{tp}(0^+/\mathbb{Q})$ is definable (e.g., $d_{x>y}(y)$ is $y \leq 0$)
- $\text{tp}(\pi/\mathbb{Q})$ is not definable by o-minimality

Quantifier-Free Heirs and Coheirs

Suppose $M \subseteq B \subset U$. Let $q(x) \in S_{\mathrm{qf}}(B)$ extend $p(x) \in S_{\mathrm{qf}}(M)$.

Definition

We say q is a *quantifier-free heir* of p iff: q "satisfies no new formulae."

Definition

We say q is a *quantifier-free coheir* of p iff: q is finitely satisfiable in M.

Fact: Quantifier-free heirs and coheirs exist.

For $a, b \in U$, TFAE:

- $\mathrm{tp}_{\mathrm{qf}}(a/Mb)$ is a quantifier-free heir of $\mathrm{tp}_{\mathrm{qf}}(a/M)$
- $\mathrm{tp}_{\mathrm{qf}}(b/Ma)$ is a quantifier-free coheir of $\mathrm{tp}_{\mathrm{qf}}(b/M)$
- for all $\phi(x, y) \in \mathcal{L}_{\mathrm{qf}}$, if $\mathcal{U} \models \phi(a, b)$, then $\mathcal{U} \models \phi(a, m)$ for some $m \in M$

Uniqueness of Quantifier-Free Heirs

Suppose $M \subseteq B \subset U$. Let $p(x) \in S_{\mathrm{qf}}(M)$.

Lemma

If p is quantifier-free definable by schema d, then $d(B)$ is the unique quantifier-free heir of p over B.

Proof: Elementarity ensures that $d(B)$ is an heir since

$$\phi(x,b) \in d(B) \ \Rightarrow \ \mathcal{U} \models d_\phi(b) \ \Rightarrow \ \mathcal{U} \models \exists y\, d_\phi(y) \ \Rightarrow \ \mathcal{M} \models \exists y\, d_\phi(y).$$

Let $q \in S_{\mathrm{qf}}(B)$ be an heir of p. In order to reach a contradiction, assume q is not $d(B)$. It follows that for some $\phi(x,y) \in \mathcal{L}_{\mathrm{qf}}$ and $b \in B$, we have

$$\neg(\phi(x,b) \leftrightarrow d_\phi(b)) \in q.$$

But since q is an heir, this implies that

$$\neg(\phi(x,m) \leftrightarrow d_\phi(m)) \in p$$

for some $m \in M$.

Uniqueness of Quantifier-Free Coheirs

Suppose $M \subseteq B \subset U$. Let $p(x) \in S_{\mathrm{qf}}(M)$.

Lemma

If every complete quantifier-free type over M is quantifier-free definable, then p has a unique quantifier-free coheir over B.

Proof: Suppose $q_1, q_2 \in S_{\mathrm{qf}}(B)$ are coheirs of p.

Let $a_1 \models q_1$, $a_2 \models q_2$, and $\phi(x, b) \in q_1$.

It follows that $\mathrm{tp}_{\mathrm{qf}}(b/Ma_1)$ and $\mathrm{tp}_{\mathrm{qf}}(b/Ma_2)$ are heirs of $\mathrm{tp}_{\mathrm{qf}}(b/M)$.

Let d be a defining schema for $\mathrm{tp}_{\mathrm{qf}}(b/M)$.

The previous lemma asserts that $\mathrm{tp}_{\mathrm{qf}}(b/Ma_i) = d(Ma_i)$ for $i = 1, 2$.

$$\begin{aligned} \phi(x, b) \in q_i \quad &\iff \quad \mathcal{U} \models \phi(a_i, b) \quad \iff \quad \phi(a_i, y) \in \mathrm{tp}_{\mathrm{qf}}(b/Ma_i) \\ &\iff \quad \mathcal{U} \models d_\phi(a_i) \quad \iff \quad d_\phi(x) \in p \end{aligned}$$

Constructing $\mathcal{M}^*$

Recall:

- $\mathcal{M}' \succ \mathcal{M}$ is $|M|^+$-saturated
- $\mathcal{L}^{\mathrm{Sh}} = \mathcal{L}_{\mathrm{ind}M'} = \{R_{\phi(x,b)} : \phi(x,b) \in \mathcal{L}(M')\}$
- $\mathcal{M}^{\mathrm{Sh}} = M_{\mathrm{ind}M'}$

Let $\mathcal{L}^* = \mathcal{L} \cup \mathcal{L}^{\mathrm{Sh}} = \mathcal{L} \cup \{R_{\phi(x,b)} : \phi(x,b) \in \mathcal{L}(M')\}$.

For each $\phi(x,b) \in \mathcal{L}(M')$, let

$$R^{\mathcal{M}^*}_{\phi(x,b)} = \phi(M,b) = \{m \in M^{|x|} : \mathcal{M}' \models \phi(m,b)\}.$$

$$\begin{tikzcd}
\mathcal{M}' \\
\curlyvee \\
\mathcal{M} & & \mathcal{M}^* \arrow[ll, "\mathcal{L}\text{-reduct}"'] \arrow[rr, "\mathcal{L}^{\mathrm{Sh}}\text{-reduct}"] & & \mathcal{M}^{\mathrm{Sh}}
\end{tikzcd}$$

Properties of $\mathcal{M}^*$

$$
\begin{tikzcd}
\mathcal{M}' \arrow[d, phantom, "\curlyvee"] & & \\
\mathcal{M} & \mathcal{M}^* \arrow[l, "\mathcal{L}\text{-reduct}"'] \arrow[r, "\mathcal{L}^{\mathsf{Sh}}\text{-reduct}"] & \mathcal{M}^{\mathsf{Sh}}
\end{tikzcd}
$$

For all $\phi(x) \in \mathcal{L}(M)$, we have

$$\mathcal{M}^* \models \phi(x) \leftrightarrow R_\phi(x).$$

Furthermore, by induction on $\mathcal{L}^*_{\mathsf{qf}}$, we conclude that for all $\psi(x) \in \mathcal{L}^*_{\mathsf{qf}}$, there exists $\theta(x) \in \mathcal{L}(M')$ such that

$$\mathcal{M}^* \models \psi(x) \leftrightarrow R_\theta(x).$$

Constructing a well-behaved $\mathcal{N}^* \succ \mathcal{M}^*$

Let $\kappa = |\mathcal{L}| + |M'|$.

Let $(\mathcal{N}', N) \succ (\mathcal{M}', M)$ be κ^+-saturated.

For each $\phi(x, b) \in \mathcal{L}(M')$, let

$$R^{\mathcal{N}^*}_{\phi(x,b)} = \phi(N, b) = \{n \in N^{|x|} : \mathcal{N}' \models \phi(n, b)\}.$$

It follows that $\mathcal{N}^* \succ \mathcal{M}^*$ is κ^+-saturated.

$$\begin{tikzcd}
\mathcal{U} \\
\mathcal{N} \arrow[u, phantom, "\curlyvee"] & \mathcal{N}^* \arrow[l, "\mathcal{L}\text{-reduct}"'] \\
\mathcal{M} \arrow[u, phantom, "\curlyvee"] & \mathcal{M}^* \arrow[l, "\mathcal{L}\text{-reduct}"'] \arrow[u, phantom, "\curlyvee"] \arrow[r, "\mathcal{L}^{\mathrm{Sh}}\text{-reduct}"] & \mathcal{M}^{\mathrm{Sh}}
\end{tikzcd}$$

Properties of $\mathcal{N}^*$

$$\begin{tikzcd}
\mathcal{U} & & & \\
\rotatebox{90}{$\succ$} & & & \\
\mathcal{N} & \mathcal{N}^* \arrow[l, "\mathcal{L}\text{-reduct}"'] & & \\
\rotatebox{90}{$\succ$} & \rotatebox{90}{$\succ$} & & \\
\mathcal{M} & \mathcal{M}^* \arrow[l, "\mathcal{L}\text{-reduct}"'] \arrow[r, "\mathcal{L}^{\mathsf{Sh}}\text{-reduct}"] & \mathcal{M}^{\mathsf{Sh}}
\end{tikzcd}$$

For all $\phi(x) \in \mathcal{L}(M)$, we have

$$\mathcal{N}^* \models \phi(x) \leftrightarrow R_\phi(x).$$

Furthermore, for all $\psi(x) \in \mathcal{L}^*_{\mathsf{qf}}$, there exists $\theta(x) \in \mathcal{L}(M')$ such that

$$\mathcal{N}^* \models \psi(x) \leftrightarrow R_\theta(x).$$

Working with Types of T^*

Let $T^* = \mathrm{Th}(\mathcal{M}^*)$.

We will use S^* when referring to type spaces of T^*.

Lemma

Each $p^*(x) \in S^*_{\mathrm{qf}}(\varnothing)$ extends uniquely to $p^*\restriction^M(x) \in S^*_{\mathrm{qf}}(M)$.

Proof: For each $\phi(x, y) \in \mathcal{L}^*_{\mathrm{qf}}$ and $m \in M^{|x|}$, we have

$$\mathcal{M}^* \models \phi(x, m) \leftrightarrow R_{y=m}(y) \wedge \phi(x, y).$$

Lemma

For each $q^*(x) \in S^*_{\mathrm{qf}}(N)$, there is a unique $q(x) \in S(N)$ such that $q^* \vdash q$.

Proof: For each $\phi(x, y) \in \mathcal{L}$ and $n \in N$, we have

$$\mathcal{N}^* \models \phi(x, n) \leftrightarrow R_\phi(x, n).$$

Types in $S^*_{\mathrm{qf}}(M)$ Are Quantifier-Free Definable

Lemma

Each $p^* \in S^*_{\mathrm{qf}}(M)$ is quantifier-free definable.

Proof: Fix $p^*(x) \in S^*_{\mathrm{qf}}(M)$ and $\psi(x, y) \in \mathcal{L}^*_{\mathrm{qf}}$. Let $a \in N$ realizes p^*.

We need to find $d_\psi(y) \in \mathcal{L}^*_{\mathrm{qf}}(M)$ whose trace in M is

$$B = \{b \in M : \mathcal{N}^* \models \psi(a, b)\}.$$

There exist $\theta(x, y) \in \mathcal{L}(M')$ such that for all $b \in M$, we have

$$\mathcal{N}^* \models \psi(a, b) \iff \mathcal{N}^* \models R_\theta(a, b) \iff \mathcal{U} \models \theta(a, b).$$

It follows that

$$B = \{b \in M : \mathcal{U} \models \theta(a, b)\}$$

and, therefore, is externally definable, so we can let d_ψ be $R_B \in \mathcal{L}^{\mathrm{Sh}}$.

T NIP $\implies$ T^* QE

Lemma

T^* has quantifier elimination if and only if for all $n < \omega$ and $p^* \in S_n^*(\varnothing)$, we have $T^* + p^*\restriction_{\mathrm{qf}} \vdash p^*\restriction_{\exists}$.

Theorem

If T is NIP, then T^* has quantifier elimination.

Proof: (Contrapositive) Suppose T^* does not have quantifier elimination.

There exists $p^*(x) \in S_{\mathrm{qf}}^*(\varnothing)$ which has more than one extension to a complete existential type over $\varnothing$.

T NIP $\implies$ T^* QE

It follows that for some $\theta(x,y) \in \mathcal{L}(M')$, both

$$p^*(x) + \exists y\, R_\theta(x,y) \qquad \text{and} \qquad p^*(x) + \neg\exists y\, R_\theta(x,y)$$

are consistent with T^*.

Let $q^*(x,y) \in S^*_{\mathrm{qf}}(\varnothing)$ be an extension of $p^*(x) + R_\theta(x,y)$.

Let $p_1^*(x) \in S^*_{\mathrm{qf}}(N)$ and $q_1^*(x,y) \in S^*_{\mathrm{qf}}(N)$ be the unique coheirs of $p^*\restriction^M$ and $q^*\restriction^M$, respectively. It follows that $p_1^*(x) = q_1^*(x,y)\rfloor_x$.

Let $r_1^*(x) \in S^*(N)$ be an extension of $p^*(x) + \neg\exists y\, R_\theta(x,y)$ which is finitely satisfiable in M. It follows that $p_1^* \subseteq r_1^*$, so $p_1^*(x) + \neg\exists y\, R_\theta(x,y)$ is finitely satisfiable in N.

T NIP $\implies$ T^* QE

Recap:

- $q_1^*(x,y) \in S^*_{\mathrm{qf}}(N)$ is finitely satisfiable in M
- $R_\theta(x,y) \in q_1^*$
- $p_1^*(x) = q_1^*\lfloor_x$
- $p_1^*(x) + \neg\exists y\, R_\theta(x,y)$ is finitely satisfiable in N

Let $p_1(x) \in S(N)$ and $q_1(x,y) \in S(N)$ be such that $p_1^* \vdash p_1$ and $q_1^* \vdash q_1$.

Claim

$q_1(x,y) + \neg R_\theta(x,y)$ is finitely satisfiable in N.

Proof of Claim: Let $a, a', b \in U$ such that

$$(a,b) \models q_1^*(x,y) \quad \text{and} \quad a' \models p_1^*(x) + \neg\exists y\, R_\theta(x,y).$$

Since $a, a' \models p_1(x)$, there exists $\sigma \in \mathrm{Aut}(\mathcal{U}/N)$ mapping $a \mapsto a'$.

Let $b' = \sigma(b)$. It follows that $(a',b') \models q_1(x,y) + \neg R_\theta(x,y)$.

T NIP $\implies$ T^* QE

Recap:

- $q_1^*(x,y) \in S^*_{\mathrm{qf}}(N)$ is finitely satisfiable in M
- $R_\theta(x,y) \in q_1^*$
- $q_1(x,y) \in S(N)$ such that $q_1^* \vdash q_1$
- $q_1(x,y) + \neg R_\theta(x,y)$ is finitely satisfiable in N

By saturation, we can construct $(a_n, b_n)_{n<\omega} \subseteq N$ so that

$$n \text{ even}: \ (a_n, b_n) \models q_1^*(x,y)\restriction_{Ma_0b_0\ldots a_{n-1}b_{n-1}}$$

$$n \text{ odd}: \ (a_n, b_n) \models q_1(x,y)\restriction_{Ma_0b_0\ldots a_{n-1}b_{n-1}} + \neg R_\theta(x,y)$$

Compactness implies that q_1 is finitely satisfiable in M, so $(a_n, b_n)_{n<\omega}$ is a coheir sequence and, as such, is $\mathcal{L}$-indiscernible over M.

Now $\mathcal{N}^* \models R_\theta(a_n, b_n)$ if and only if n is even, so $\mathcal{U} \models \theta(a_n, b_n)$ if and only if n is even. Thus, T is IP. $\square$

Active Research

Open Questions:

- In general, what conditions are sufficient for $M_{\text{ind}A}$ to have QE?
- If $\mathcal{I}$ is a Morley sequence of an M-invariant type, does $M_{\text{ind}\,\mathcal{I}}$ have QE?

Closed Question:

- If $\mathcal{I}$ is a Morley sequence of an M-invariant type p and $p^{(\omega)}$ is both an heir and a coheir of its restriction to M, does $M_{\text{ind}\,\mathcal{I}}$ have QE?

  YES (Simon, 2013)